

MAGIDOR-MALITZ REFLECTION

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ABSTRACT. In this paper we investigate the consequences and consistency of the downward Löwenheim-Skolem theorem for extension of the first order logic by the Magidor-Malitz quantifier. We derive some combinatorial results and improve the known upper bound for the consistency of Chang's conjecture at successors of singular cardinals.

1. INTRODUCTION

The downward Löwenheim-Skolem theorem states that any model M of cardinality λ and infinite cardinal $\kappa < \lambda$ has an elementary submodel $N \prec M$ with cardinality κ . This is a reflection theorem, i.e. it states that whenever there is some structure M of uncountable cardinality that satisfies some first order sentence, then there is already some small submodel satisfying the same sentence.

While this theorem is extremely useful by itself, in many cases one would like to reflect downwards second order properties of the model M . This motivation lies at the basis of the diverse theory of reflection of second order properties. While the Löwenheim-Skolem theorem shows that there is no set theoretical restrictions on the reflection of first order properties, the next theorem, due to Magidor, shows that reflecting all the second order theory is possible only above a supercompact cardinal:

Theorem 1 (Magidor). [7] *Assume that there is κ such that for every model M with countable language, there is $N \prec_{\mathcal{L}^2} M$, $|N| < \kappa$, then there is a supercompact cardinal $\leq \kappa$.*

Indeed, even full Π_1^1 -reflection implies the existence of a supercompact cardinal.

In order to consistently have a reflection principle for a fragment of second order logic at accessible cardinals, we must settle for less. A classical example in this vein is Chang's conjecture:

Definition 2. Let $\kappa, \lambda, \mu, \nu$ be cardinals, $\kappa > \lambda$, $\mu > \nu$. We say that $(\kappa, \lambda) \twoheadrightarrow (\mu, \nu)$ is for every model M of the countable language \mathcal{L} with distinct unary predicate A such that $|M| = \kappa$ and $|A| = \lambda$ there is an elementary submodel $N \prec M$ such that $|N| = \mu$, $|A \cap N| = \nu$.

Chang's conjecture is a natural strengthening of the model theoretical two cardinals theorems of Vaught and Chang. There is an extensive literature about Chang's conjecture for various parameters. See, for example, section 7.3 in [1].

The following quantifier was defined by Menachem Magidor and Jerome Malitz in [8]:

Definition 3 (Magidor-Malitz Quantifier). Let M be a model in the language \mathcal{L} . For a formula $\varphi(x_0, x_1, \dots, x_{n-1}, p_0, \dots, p_{m-1})$, we write

$$M \models Q^n x_0, \dots, x_{n-1} \varphi(x_0, x_1, \dots, x_{n-1}, p_0, \dots, p_{m-1})$$

if there is a set $A \subseteq M$ with $|A| = |M|$ such that

$$\forall a_0, a_1, \dots, a_{n-1} \in A, M \models \varphi(a_0, \dots, a_{n-1}, p_0, \dots, p_{m-1}).$$

We write $M \prec_{Q^n} N$ if M is elementary submodel of N with respect to first order logic enriched with the quantifier Q^n . We write $M \prec_{Q^{<\omega}} N$ if $M \prec_{Q^n} N$ for all $n < \omega$.

Let $\varphi(x_0, \dots, x_{n-1}, p)$ be a formula with free variables x_0, \dots, x_{n-1} and parameter p (for simplicity, we assume that there is only one parameter). A set $A \subseteq M$ is called φ -cube if for all $a_i \in A$, $M \models \varphi(a_0, \dots, a_{n-1}, p)$.

The Magidor-Malitz quantifier enables us to express some second order properties of the model. We are interested in the possibility of *reflection* of $Q^{<\omega}$, which means informally that for many pairs of models $M \prec N$ we have $M \prec_{Q^{<\omega}} N$. For example, Q^1 is the Mostowski quantifier with the $|M|$ interpretation. Instances of Q^1 reflection are equivalent to several instances of Chang's conjecture (see lemma 5).

In section 2 we will define the $Q^{<\omega}$ analogue for Chang's conjecture and derive some reflection principles from it. In section 3 we will investigate the large cardinals which imply $Q^{<\omega}$ reflection and prove consistency results about some cases of $Q^{<\omega}$ reflection at small cardinals.

2. COMBINATORIAL CONSEQUENCES OF $Q^{<\omega}$ REFLECTION

In this section we analyse the relationship between the reflection of the Magidor-Malitz quantifier, $Q^{<\omega}$, and some square like principles.

The Magidor-Malitz quantifier allows us to access some of the second order properties of the model. As we will see, the downward Löwenheim-Skolem theorem for the quantifiers Q^n is a strong reflection principle, yet it consistently holds for some pairs of small cardinals (assuming the consistency of large cardinals).

Definition 4. Let λ, μ be cardinals. $\lambda \xrightarrow[Q^{<\omega}]{\eta} \mu$ iff for every model of cardinality λ , over a language of cardinality η , there is a $Q^{<\omega}$ -elementary submodel of cardinality μ . When $\eta = \aleph_0$ we write $\lambda \xrightarrow[Q^{<\omega}]{} \mu$. Similarly, $\lambda \xrightarrow[Q^n]{} \mu$ iff for every model of cardinality λ there is a Q^n elementary submodel of cardinality μ .

$\lambda \xrightarrow[Q^{<\omega}]{} < \mu$ abbreviates the assertion that for every model of cardinality λ there is a $Q^{<\omega}$ -elementary submodel of cardinality less than μ .

Let us recall that a weak instance of the reflection principle $\lambda \xrightarrow[Q^{<\omega}]{} \mu$ is equivalent to Chang's conjecture:

Lemma 5. Let $\mu < \lambda$ be cardinals. $\lambda^+ \xrightarrow[Q^1]{} \mu^+$ iff $(\lambda^+, \lambda) \twoheadrightarrow (\mu^+, \mu)$.

Proof. Let us assume that $\lambda^+ \xrightarrow[Q^1]{} \mu^+$. Let (M, A) be a model of type (λ^+, λ) .

Assume, without loss of generality, that:

- (1) A is a predicate in the language
- (2) There is a well ordering \leq^* on M with order type λ^+
- (3) For every $a \in M$ there is a definable surjection from A onto the elements that are smaller than a in \leq^* .

Let $N \prec_{Q^1} M$ be a elementary submodel of cardinality μ^+ . We claim that $A^N = A \cap N$ has cardinality μ .

$M \models \neg Q^1 x \in A$ (since $|A| = \lambda < |M| = \lambda^+$) and therefore $N \models \neg Q^1 x \in A$, so $|A^N| \leq \mu$. On the other hand, by elementarity for every $a \in N$ there is a surjection from A^N onto $\{b \in N \mid b \leq^* a\}$ so $|A|$ cannot be strictly smaller than μ .

On the other hand, assume that $(\lambda^+, \lambda) \twoheadrightarrow (\mu^+, \mu)$. By enriching the language, we may assume that for every formula $\phi(x, b)$ there is a function symbol f_ϕ such that $\{f_\phi(x, b) \mid x \in M\} = \{y \in M \mid M \models \phi(y, b)\} = \phi(M, b)$. Moreover, we may

choose f_ϕ in a way that if $\phi(M, b)$ has cardinality $\leq \lambda$ then $x \rightarrow f_\phi(x, b)$ is onto already when restricted to A . Otherwise, we assume that $x \rightarrow f_\phi(x, b)$ is one to one.

Let $N \prec M$ be an elementary submodel with $|A^N| = \mu$. Let us look at the formula $Q^1 x \phi(x, b)$. If it holds in M , then $f_\phi(x, b)$ enumerates the set of witnesses and when restricting this function to N , we get a one to one function from N such that $N \models \forall x \phi(f_\phi(x, b), b)$. Thus $|\{x \in N \mid \phi(x, b)\}| = |N|$. On the other hand, if $\neg Q^1 x \phi(x, b)$ then

$$M \models \forall x \phi(x, b) \rightarrow \exists a \in A, x = f_\phi(a, b).$$

Therefore, N satisfies the same formula and $|\{x \in N \mid \phi(x, b)\}| \leq |A^N|$. \square

The previous lemma shows that the reflection principle $\mu^+ \xrightarrow{Q^{<\omega}} \kappa$ is rather strong. For example, since $(\aleph_{\omega+1}, \aleph_\omega) \not\xrightarrow{Q^{<\omega}} (\aleph_n, \aleph_{n-1})$ for all $n \geq 4$, we conclude that $\aleph_{\omega+1} \not\xrightarrow{Q^{<\omega}} \aleph_n$ for all $n \geq 4$.

The proof of the lemma shows that if $\lambda^+ \xrightarrow{Q^1} \mu$ then μ must be a successor cardinal and in particular regular. Similarly, if $\lambda^+ \xrightarrow{Q^1} < \kappa$ then we may assume always that the cardinalities of the elementary submodels are successor cardinals.

Let us start with the following useful observation which shows that models that are obtained from Chang's conjecture can be assumed to have a specific order type.

Lemma 6. Assume $\lambda \xrightarrow{Q^1} \mu$. Then for every model \mathcal{A} on λ there is an elementary submodel, \mathcal{B} , such that $\text{otp } \mathcal{B} = \mu$.

Proof. Assume that the ordinal order in part of the language of \mathcal{A} . Let us reflect the statement:

$$\forall \alpha \neg Q^1 \beta, \beta < \alpha$$

from \mathcal{A} into \mathcal{B} . Observe that for every $\alpha \in \mathcal{B}$, $\text{otp } \mathcal{B} \cap \alpha < \mu$. Therefore $\text{otp } \mathcal{B}$ is an increasing union of μ ordinals all strictly smaller than μ . \square

Let us recall the definition of $\square(\kappa)$.

Definition 7. Let $\mathcal{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ be a sequence of closed sets such that:

- (1) $\sup C_\alpha = \alpha$ for all limit ordinal α .
- (2) If $\beta \in \text{acc } C_\alpha$ then $C_\alpha \cap \beta = C_\beta$.
- (3) There is no club D such that $\forall \alpha \in \text{acc } D, D \cap \alpha = C_\alpha$.

Then \mathcal{C} is called a $\square(\kappa)$ sequence. We say that $\square(\kappa)$ holds if there is a $\square(\kappa)$ sequence.

This definition, due to Todorcevic or Velicovic, is pivotal in the research of reflection properties, in particular when dealing with Π_1^1 statements. See [10] for extensive review.

Theorem 8. Let us assume $\kappa \xrightarrow{Q^2} \mu$ where μ is regular. Then $\square(\kappa)$ fails.

Proof. Let $\mathcal{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ be a coherent \mathcal{C} -sequence, i.e. sequence that satisfies the first two conditions in Definition 7. We will show that there is a *thread*, namely a club D such that for every $\alpha \in \text{acc } D$, $D \cap \alpha = C_\alpha$.

Let $\mathcal{A} \prec H(\chi)$ for some large enough χ , with $\kappa \subseteq \mathcal{A}$, $\mathcal{C} \in \mathcal{A}$ and $|\mathcal{A}| = \kappa$.

Let $\mathcal{B} \prec_{Q^2} \mathcal{A}$ with $|\mathcal{B}| = \mu$. Since μ is a regular cardinal, $\sup(\mathcal{B} \cap \kappa) = \rho$, cf $\rho = \mu$ (by Lemma 6).

Let us look at $\delta \in \text{acc } C_\rho \cap \text{acc } (\mathcal{B} \cap \kappa)$ below ρ . Let $\beta = \min(\mathcal{B} \cap \kappa \setminus \delta)$. β is well defined, since $\delta < \rho = \sup \mathcal{B} \cap \kappa$. Let us show that $\delta \in \text{acc } C_\beta$.

Let $\alpha \in \mathcal{B} \cap \beta$. By the minimality of β , $\alpha < \delta$. $\gamma = \min C_\beta \setminus \alpha \in \mathcal{B}$. By the same argument, $\gamma < \delta$. Since δ is an accumulation point of ordinals in \mathcal{B} , we get that the set of ordinals in C_β below δ is unbounded, as required.

Since \mathcal{C} is coherent, $C_\delta \leq C_\beta$.

Now, let $\delta < \delta'$ be in $\text{acc } C_\rho \cap \text{acc}(\mathcal{B} \cap \kappa)$. Let $\beta = \min(\mathcal{B} \cap \kappa \setminus \delta)$, $\beta' = \min(\mathcal{B} \cap \kappa \setminus \delta')$. We claim that $C_\beta \leq C_{\beta'}$, since otherwise there is some $\gamma < \beta$ such that $\gamma \in C_\beta \triangle C_{\beta'}$. Such γ must appear in \mathcal{B} (by elementarity), so it is smaller than δ . But $C_\delta \leq C_{\delta'} \leq C_{\beta'}$ - a contradiction.

We conclude that

$$\mathcal{B} \models Q^2 \alpha, \beta \ C_\alpha \leq C_\beta$$

Therefore, \mathcal{A} contains a set of cardinality κ , I , of elements which are compatible in \mathcal{C} . $D = \bigcup_{\alpha \in I} C_\alpha$ is a thread. \square

2.1. The tree property at successor of singular.

Theorem 9. *Let μ be a singular cardinal and assume that $\langle \kappa_i \mid i < \text{cf } \mu \rangle$ is cofinal in μ , $\kappa_0 \geq \text{cf } \mu$, $\text{cf } \kappa_i = \kappa_i$ for all i . If for every $i < \text{cf } \mu$, $\mu^+ \xrightarrow[Q^{<\omega}]{\kappa_i} \kappa_{i+1}$ then the tree property holds at μ^+ .*

Proof. We prove the theorem by two steps. First we apply $Q^{<\omega}$ -reflection in order to find for a given Aronszajn tree T a narrow subsystem. At this step we will use $\mu^+ \xrightarrow[Q^{<\omega}]{\text{cf } \mu} \lambda$ for some regular $\lambda < \mu^+$. Then we pick i large enough so that κ_i is larger than the width of the system and use $\mu^+ \xrightarrow[Q^2]{\kappa_i} \kappa_{i+1}$ in order to get a branch through the narrow system.

Let T be an Aronszajn tree and assume without loss of generality that $T = \langle \mu^+ \times \mu, \leq_T \rangle$, i.e. that the α -th level of T is given by $\{\alpha\} \times \mu$. Let \mathcal{A}_0 be an elementary substructure of $H(\chi)$ for some large enough χ , such that $|\mathcal{A}_0| = \mu^+$, $\mu^+ \subseteq \mathcal{A}_0$ and $T \in \mathcal{A}_0$.

Let \mathcal{B}_0 be a $Q^{<\omega}$ -elementary substructure of \mathcal{A}_0 , containing $\text{cf } \mu$ such that the order type of $\mathcal{B}_0 \cap \mu^+$ has cofinality above $\text{cf } \mu$. We may assume, without loss of generality, that $\{\kappa_i \mid i < \text{cf } \mu\} \subseteq \mathcal{B}_0$. Let $\Delta = \mathcal{B}_0 \cap \mu^+$ - the set of levels that appear in \mathcal{B}_0 . Let $\delta = \sup \Delta$ and let us consider the branch below $\langle \delta, 0 \rangle$. Since T is a tree, for every $\alpha \in \Delta$ there is $\zeta < \mu$ such that $\langle \alpha, \zeta \rangle \leq_T \langle \delta, 0 \rangle$.

Since μ is singular and $\text{cf } \mu \subseteq \mathcal{B}_0$, there is some $i \in \mathcal{B}_0$ such that $\zeta < \kappa_i$. If α, β are both in Δ , and $\langle \alpha, \zeta \rangle, \langle \beta, \xi \rangle \leq_T \langle \delta, 0 \rangle$ then $\langle \alpha, \zeta \rangle \leq_T \langle \beta, \xi \rangle$. If we assume that $\zeta, \xi < \kappa_i$ then by elementarity there are $\tilde{\zeta}, \tilde{\xi} \in \mathcal{B}_0 \cap \kappa_i$ such that $\langle \alpha, \tilde{\zeta} \rangle \leq_T \langle \beta, \tilde{\xi} \rangle$.

Since the cofinality of δ is larger than $\text{cf } \mu$, there is $i < \text{cf } \mu$ such that \mathcal{B}_0 satisfies the $Q^{<\omega}$ formula:

$$Q^2 \alpha, \beta \exists \zeta, \xi < \kappa_i \langle \alpha, \zeta \rangle \leq \langle \beta, \xi \rangle$$

By $Q^{<\omega}$ -elementarity there is some subset $I \subseteq \mathcal{A}_0$ with cardinality μ^+ such that every element of I is an ordinal and the elements of I satisfy the same compatibility relation. Therefore, we can define a narrow system on $I \times \kappa_i$ (with only single relation), by the restriction of the tree T to this set.

Let us show now that the narrow system property follows from the assumption that $\mu^+ \xrightarrow[Q^2]{\kappa_i} \kappa_{i+1}$ for cofinally many regular $\kappa_i < \mu$.

Lemma 10. *Let μ be a singular cardinal and assume that for cofinally many regular cardinals $\kappa < \mu$, there is a regular cardinal λ , such that $\kappa < \lambda < \mu$ and $\mu^+ \xrightarrow[Q^{<\omega}]{\kappa} \lambda$. Then the narrow system property holds at μ^+ .*

Proof. Let \mathcal{S} be a narrow system with height μ^+ and less than κ many order relations and elements in each level.

Let \mathcal{A}_1 be an elementary substructure of $H(\chi)$ containing all ordinals in μ^+ and \mathcal{S} . Let us pick a $Q^{<\omega}$ -elementary substructure of $\mathcal{A}_1, \mathcal{B}_1$ of cardinality strictly large than κ , containing all ordinals below κ . Let $\delta = \sup \mathcal{B}_1 \cap \mu^+$ and let us pick some element $\epsilon \in I \setminus \delta$. Since \mathcal{S} is a narrow system, for every $\alpha \in \mathcal{B}_1$ there are $\zeta, \xi < \kappa$ and index $i < \kappa$ such that $\langle \alpha, \zeta \rangle \leq_i \langle \epsilon, \xi \rangle$. By lemma Lemma 6, we may assume that $\text{otp } \mathcal{B}_1$ is regular and therefore, for unbounded many ordinals below ϵ in \mathcal{B}_1 the tuple (ζ, ξ, i) is constant. Therefore, for some $\zeta_*, i_* < \kappa$, \mathcal{B}_1 satisfies:

$$Q^2 \alpha, \beta, \langle \alpha, \zeta_* \rangle \leq_{i_*} \langle \beta, \zeta_* \rangle$$

The same holds in \mathcal{A}_1 , and therefore there is a branch in \mathcal{S} . \square

Applying Lemma 10 on $T \restriction I$, we obtain a cofinal branch through $T \restriction I, b'$. The set $\{s \in T \mid \exists s \in b, t \leq s\}$ is a cofinal branch through T . \square

The assumptions of Theorem 9 and Lemma 10 can be weakened to the assumption that for every model \mathcal{A} of cardinality μ^+ over language of cardinality $\eta < \mu$ there is some regular cardinal $\kappa < \mu$ and an Q^2 -elementary submodel \mathcal{B} of cardinality κ . This type of reflection principle follows from large cardinals at the level of supercompact (see Theorem 16). It is unclear whether one can derive this kind of reflection from strongly compact cardinals. In fact, it is unclear even if one can derive some instances of Chang's conjecture from strongly compact cardinals.

The assumption that $\mu^+ \xrightarrow[Q^{<\omega}]{} \kappa$ for cofinally many $\kappa < \mu$ seems to be stronger than the narrow system property. For example, it cannot hold for $\mu = \aleph_\omega$.

3. CONSISTENCY RESULTS

This section is dedicated to the derivation of some consistency results regarding the reflection principles that were defined above.

3.1. Chang's conjecture at $\aleph_{\omega+1}$. We begin this section with two theorems about the consistency of Chang's conjecture at successors of singular cardinals.

Definition 11 (Jensen). [9] A cardinal κ is $(+\alpha)$ *subcompact* if for every $A \subseteq H(\kappa^{+\alpha})$ there is $\rho < \kappa$ and $B \subseteq H(\rho^{+\alpha})$ and elementary embedding

$$j: \langle H(\rho^{+\alpha}), \in, \rho, B \rangle \rightarrow \langle H(\kappa^{+\alpha}), \in, \kappa, A \rangle$$

where ρ is the critical point of j . A cardinal κ is *subcompact* if it is $(+1)$ -subcompact.

In order to get a general feeling about the place of this type of cardinals in the large cardinals hierarchy, let us remark that if κ is $\kappa^{+\omega+1}$ supercompact and $\kappa^{+\omega}$ is strong limit then κ is $(+\omega + 1)$ subcompact and have a normal measure concentrating on $(+\omega)$ -subcompact cardinals below it. On the other hand, if a cardinal κ is $(+\omega + 1)$ subcompact then it is κ^{+n} supercompact for every n and has a normal measure concentrating on cardinals ρ which are ρ^{+n} supercompact.

Theorem 12. Let κ be $(+\omega + 1)$ -subcompact cardinal, $\kappa^{+\omega}$ strong limit. There is $\rho < \kappa$ such that $(\kappa^{+\omega+1}, \kappa^{+\omega}) \twoheadrightarrow (\rho^{+\omega+1}, \rho^{+\omega})$.

Proof. Let $\mu = \kappa^{+\omega}$. Assume otherwise, and let us pick for every $\rho < \kappa$ a function $f_\rho: (\mu^+)^{<\omega} \rightarrow \mu^+$ such that for all $R \subseteq \mu^+$ of cardinality $\rho^{+\omega+1}$, $|f_\rho''[R]^{<\omega} \cap \mu| \neq \rho^{+\omega}$.

Let us code this sequence of functions as a subset of $H(\kappa^{+\omega+1})$, A .

Let $j: \langle H(\rho^{+\omega+1}), B \rangle \rightarrow \langle H(\kappa^{+\omega+1}), A \rangle$ be an elementary embedding as in the definition of $(+\omega + 1)$ -subcompactness. B codes a sequence of functions from

$(\rho^{+\omega+1})^{<\omega}$ to $\rho^{+\omega+1}$, $\langle g_\eta \mid \eta < \rho \rangle$, witnessing the failure of Chang's conjecture. Note that $\rho^{+\omega}$ is strong limit.

Let us look at f_ρ . Let $R = j''\rho^{+\omega+1} \in H(\kappa^{+\omega+1})$. By our assumptions, $|f_\rho''[R]^{<\omega} \cap \mu| > \rho^{+\omega}$. Let n be the first ordinal such that $|f_\rho''[R]^{<\omega} \cap \kappa^{+n}| = \rho^{+\omega+1}$.

Since $\text{cf } \rho^{+\omega} = \omega$ and $\rho^{+\omega+1}$ is regular, it is impossible that $n = \omega$, so n is a natural number.

Let $\langle \vec{\alpha}_\xi \mid \xi < \rho^{+\omega+1} \rangle$ be a sequence of elements in $(\rho^{+\omega+1})^{<\omega}$, and assume that $\langle f_\rho(j(\vec{\alpha}_\xi)) \mid \xi < \rho^{+\omega+1} \rangle$ is strictly increasing sequence of ordinals below κ^{+n} .

By elementarity, for every $\xi \neq \xi'$,

$$\langle g_\eta(\vec{\alpha}_\xi) \cap \rho^{+n} \mid \eta < \rho \rangle \neq \langle g_\eta(\vec{\alpha}_{\xi'}) \cap \rho^{+n} \mid \eta < \rho \rangle$$

Otherwise, for every $\tilde{\rho} < \kappa$ we would get that

$$f_{\tilde{\rho}}(j(\vec{\alpha}_\xi)) \cap \kappa^{+n} = f_{\tilde{\rho}}(j(\vec{\alpha}_{\xi'})) \cap \kappa^{+n}$$

and evaluating at $\tilde{\rho} = \rho$ we get a contradiction.

But the number of possible sequences of this form is $(\rho^{+n})^\rho < \rho^{+\omega}$ - a contradiction. \square

Theorem 13. *Let κ be a $(+\omega + 1)$ -subcompact cardinal, $\kappa^{+\omega}$ strong limit. There is $\rho < \kappa$ such that forcing with $\text{Col}(\omega, \rho^{+\omega}) \times \text{Col}(\rho^{+\omega+2}, < \kappa)$ forces the instance of Chang's conjecture $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$.*

Proof. Assume otherwise, and let us pick for all $\rho < \kappa$ a $\text{Col}(\omega, \rho^{+\omega}) \times \text{Col}(\rho^{+\omega+2}, < \kappa)$ -name for a function $f_\rho: (\mu^+)^{<\omega} \rightarrow \mu^+$, witnessing the failure of Chang's conjecture in the generic extension. Note that we can still code this set of names as a subset of $H(\kappa^{+\omega+1})$, A .

Let $j: \langle H(\rho^{+\omega+1}), \in, B \rangle \rightarrow \langle H(\kappa^{+\omega+1}), \in, A \rangle$ be the subcompact embedding. As in the previous proof, we denote by \dot{g}_η the names of the functions coded by B . Let us look at f_ρ and the forcing $\text{Col}(\omega, \rho^{+\omega}) \times \text{Col}(\rho^{+\omega+2}, < \kappa)$.

Let $R = j''\rho^{+\omega+1}$.

By the assumption, $\Vdash |f_\rho''[R]^{<\omega} \cap \mu| = \rho^{+\omega+1}$ and by the same argument as before, there is some condition (p_0, p_1) and a minimal ordinal n such that

$$(p_0, p_1) \Vdash |f_\rho''[R]^{<\omega} \cap \kappa^{+n}| = \rho^{+\omega+1}.$$

Since $\rho^{+\omega+1}$ is still regular after the forcing, n must be finite.

Let $\{\dot{a}_\xi \mid \xi < \rho^{+\omega+1}\}$ be a sequence of names of finite sequences of ordinals below $\rho^{+\omega+1}$ such that it is forced by the empty condition that $f_\rho(j(\dot{a}_\xi)) < f_\rho(j(\dot{a}_{\xi'})) < \kappa^{+n}$ for all $\xi < \xi'$.

Since the $\text{Col}(\rho^{+\omega+2}, < \kappa)$ is $\rho^{+\omega+2}$ -closed, we can find a condition that below it the value of \dot{a}_ξ is determined only by the first coordinate for all $\xi < \rho^{+\omega+1}$.

There are $\rho^{+\omega}$ many conditions in $\text{Col}(\omega, \rho^{+\omega})$. Therefore, there is a single condition p and a set of size $\rho^{+\omega+1}$ of finite sequences such that p decides on the value of all of them. Let $\{b_\xi \mid \xi < \rho^{+\omega+1}\}$ be an enumeration of this set.

Back in $H(\rho^{+\omega+1})$, for every $\xi < \xi'$ there is $\eta < \rho$ and a condition q that forces $g_\eta(b_\xi) < g_\eta(b_{\xi'}) < \rho^{+n}$.

This defines a coloring $[\rho^{+\omega+1}]^2 \rightarrow V_\rho \times \rho^{+n}$. Let us restrict the coloring to the first $(2^{\rho^{+n}})^+$ sequences. By Erdős-Rado theorem, there is a homogeneous set of cardinality ρ^{+n+1} , H . Let (η, q) be the color of all pairs in H . So for every $\xi < \xi'$ in H ,

$$q \Vdash_{\text{Col}(\omega, \eta^{+\omega}) \times \text{Col}(\eta^{+\omega+2}, < \rho)} \dot{g}_\eta(b_\xi) < \dot{g}_\eta(b_{\xi'}) < \rho^{+n}$$

and in particular, after forcing with $\text{Col}(\omega, \eta^{+\omega}) \times \text{Col}(\eta^{+\omega+2}, < \rho)$ below the condition q , there is a set of order type ρ^{+n+1} below ρ^{+n} , which is impossible. \square

Assuming the consistency of a $(+\omega+1)$ -subcompact cardinal, it is consistent that κ is $(+\omega+1)$ -subcompact and $\square(\kappa^{+\omega+1})$ holds (yet $\square_{\kappa^{+\omega}}^*$ and even the approachability property must fail, by Theorem 12). Therefore in the model of Theorem 13 we may have $\square(\aleph_{\omega+1})$ and therefore $\aleph_{\omega+1} \not\rightarrow_{Q^{<\omega}} \aleph_1$.

The assumption in both Theorem 12 and Theorem 13 is slightly below the assumption of κ being $\kappa^{+\omega+1}$ -supercompact.

Question. Is Chang's conjecture $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ consistent assuming the consistency of a strongly compact cardinal?

3.2. MM submodels. In this subsection we investigate which large cardinal assumptions imply the $Q^{<\omega}$ -reflection. We first deal with the case $\lambda \xrightarrow{Q^{<\omega}} \kappa$ for λ successor cardinal.

3.2.1. Π_1^1 Subcompact cardinals. The following large cardinal notion was defined by Neeman and Steel in [9]. We will use a slightly different notation than the one that is used in this paper.

Definition 14. A cardinal κ is Π_1^1 -($+\alpha$)-subcompact, if for every $A \subseteq H(\kappa^{+\alpha})$ and Π_1^1 -statement ϕ such that $\langle H(\kappa^{+\alpha}), \kappa, \in, A \rangle \models \phi$ there is $\rho < \kappa$ and $B \subseteq H(\rho^{+\alpha})$ such that $\langle H(\rho^{+\alpha}), \in, \rho, B \rangle \models \phi$.¹

In order to get a general feeling above the place of Π_1^1 -subcompact cardinals in the large cardinals hierarchy, we remark that a Π_1^1 -($+0$)-subcompact is weakly compact, while $(+0)$ -subcompact cardinal is inaccessible cardinal².

Lemma 15. Let κ be a Π_1^1 -($+\alpha$)-subcompact cardinal, $\alpha < \kappa$. Then there is a stationary subset of κ of $(+\alpha)$ -subcompact cardinals.

Proof. Let us note that being a $(+\alpha)$ -subcompact cardinal is equivalent to a seemingly weaker assumption in which we assume only that j is Σ_1 elementary relative to additional predicate (by coding the full elementary diagram). Using this interpretation, the statement " κ is $(+\alpha)$ -subcompact" is Π_1^1 -statement and therefore reflects downwards. By adding a predicate C for a club, we obtain the desired result. \square

Theorem 16. Let κ be Π_1^1 -($+\alpha$) subcompact, $\alpha < \kappa$ successor ordinal and assume that $|H(\kappa^{+\alpha})| = \kappa^{+\alpha}$. Then $\kappa^{+\alpha} \xrightarrow{Q^{<\omega}} < \kappa$.

Proof. Let $\lambda = \kappa^{+\alpha}$.

Let \mathcal{A} be an algebra on λ . \mathcal{A} can be coded by a single predicate on $H(\lambda)$, A . Moreover, we assume that A codes also the truth predicate of \mathcal{A} and that the language of \mathcal{A} contains some bijection between $H(\lambda)$ and λ .

For every formula φ in the language of the model \mathcal{A} with the $Q^{<\omega}$ quantifiers we enrich the language of \mathcal{A} by adding one function symbol. For φ of the form $Q^n x_0, \dots, x_{n-1} \psi(x_0, \dots, x_{n-1}, b)$, let us add the function symbol $F_\varphi: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ and interpret it such that whenever $\mathcal{A} \models \varphi(b)$ (where $b \in \mathcal{A}$), the function $x \rightarrow F_\varphi(x, b)$ is one to one and its image, I , witness φ . Namely, $\forall x_0, \dots, x_{n-1} \in \mathcal{A}$, $\mathcal{A} \models \psi(F_\varphi(x_0, b), \dots, F_\varphi(x_{n-1}, b), b)$. We will assume that the truth predicate A contains also the truth value of all formulas in the enriched language.

Let Φ be the Π_1^1 sentence: For every $X \subseteq H(\lambda)$ one of the following cases holds:

¹In [9], Neeman and Steel denoted by Π_1^2 -subcompact the large cardinal notion that is denote here by Π_1^1 -($+1$)-subcompact.

²This depends on the precise definition of $H(\chi)$ for singular χ . If we define $H(\chi)$ to be the collection of sets of cardinality $< \chi$ such that every member of them belongs to $H(\chi)$ we conclude that $(+0)$ -subcompact cardinal is Mahlo.

- (1) There is $x \in X$ which is not of the form $\langle \phi, p, x \rangle$ where ϕ is (a Gödel number of) $Q^{<\omega}$ formula, $x, p \in H(\lambda)$.
- (2) There is a pair of elements $\langle \phi, p, x \rangle, \langle \phi', p', x' \rangle \in X$ with $\langle \phi, p \rangle \neq \langle \phi', p' \rangle$
- (3) $\mathcal{A} \models \phi(p)$
- (4) $\phi = Q^n x_0, \dots, x_{n-1} \varphi(x_0, \dots, x_{n-1}, p)$ and there are a_0, \dots, a_{n-1} such that $\forall i < n, \langle \phi, p, a_i \rangle \in X$ and $\mathcal{A} \models \neg \varphi(a_0, \dots, a_{n-1})$.

Where all the truth values are evaluated using the truth predicate and the variation of arity in the last case is solved using standard coding.

Let ρ, B and $j: \langle H(\rho^{+\alpha}), \in, B \rangle \rightarrow \langle H(\kappa^{+\alpha}), \in, A \rangle$ witness the assumption that κ is Π_1^1 -($+\alpha$)-subcompact relative to the Π_1^1 formula Φ .

Let us claim that $\mathcal{B} = j'' H(\rho^{+\alpha})$ is a $Q^{<\omega}$ elementary substructure of \mathcal{A} .

We need to show that for a $Q^{<\omega}$ -formula φ , and $b \in \mathcal{B}$, $\mathcal{B} \models \varphi(b)$ if and only if $\mathcal{A} \models \varphi(b)$.

We prove the claim by induction on the complexity of φ . Elementarity for first order quantifiers and connectives follows from the elementarity of j . Let us assume that φ has the form $Q^n x_0, \dots, x_{n-1} \psi(x_0, \dots, x_{n-1}, y)$, and that absoluteness holds for all formulas in the complexity level of ψ .

If $\mathcal{A} \models \varphi(b)$, then $g(x) = F_\varphi(x, b)$ enumerates some set I such that for every $a_0, \dots, a_{n-1} \in I$, $\mathcal{A} \models \psi(a_0, \dots, a_{n-1})$. By elementarity of j , when restricting g to elements of \mathcal{B} its range will be a subset of \mathcal{B} which is a ψ -block of cardinality $|\mathcal{B}|$.

Let us assume that $\mathcal{B} \models \varphi$. Then

$$\langle H(\rho^{+\alpha}), \in, B \rangle \models \exists I \text{ unbounded } \forall a_0, \dots, a_{n-1} \in I, \mathcal{B} \models \psi(a_0, \dots, a_{n-1}, b)$$

By Π_1^1 elementarity, (choosing X to be $\{\langle \varphi, b, x \rangle \mid x \in I\}$) the same holds at $\langle H(\kappa^{+\alpha}), \in, A \rangle$, and therefore $\mathcal{A} \models \varphi$. \square

Theorem 16 is parallel to Theorem 12. Unfortunately, we do not know how to generalize the stronger result of Theorem 13. For successors of regulars and target \aleph_1 , subsection 3.4 gives some partial results.

3.2.2. Inaccessible cardinals. For inaccessible cardinals, the consistency strength seems to be lower.

Theorem 17. *Let κ be Ramsey cardinal. Then for every $\omega < \mu < \kappa$, $\kappa \xrightarrow[Q^{<\omega}]{} \mu$.*

Proof. Let \mathcal{A} be an algebra on κ . Let I be a set on indiscernibles for \mathcal{A} and let \mathcal{B} be the substructure of \mathcal{A} generated by the first μ indiscernibles.

As in the previous proof, in order to show that $\mathcal{B} \prec_{Q^{<\omega}} \mathcal{A}$, we may enrich the language of \mathcal{A} by functions that produce witnesses for all $Q^{<\omega}$ formulas that hold in \mathcal{A} and show only that for every $Q^{<\omega}$ formula $\varphi = Q^n x_0, \dots, x_n \psi$, if $\mathcal{B} \models \varphi$ then $\mathcal{A} \models \varphi$.

Let $J \subseteq \mathcal{B}$ be any set of cardinality μ . Every element in $a \in J$ can be represented as $f(\alpha_0, \dots, \alpha_{m-1})$ where f is one of the Skolem functions of \mathcal{A} and $\alpha_i \in I$.

Since there are only countably many Skolem functions, f , there is some fixed f_* and uncountable subset of J , K such that for every $a \in K$ there are indiscernibles $\alpha_0, \dots, \alpha_{m-1}$ such that $a = f_*(\alpha_0, \dots, \alpha_{m-1})$. Moreover, if γ is the maximal indiscernible that appears in the description of the parameters of the formula ψ , we may assume that for all $a \in K$, $a = f_*(\alpha_0, \dots, \alpha_{m-1})$ and the set $\{\alpha_i \mid i < m, \alpha_i \leq \gamma\}$ is fixed.

By Δ -system arguments, there is some finite set $r \in I^k$ and a set $J \subseteq I^{m-k}$ such that $|J| = \mu$ and $f_*(r \frown s) \in I$ for all $s \in J$. Moreover, we may assume that for all $s \neq s'$, $f_*(r \frown s) \neq f_*(r \frown s')$ and $\max s < \min s'$ or $\min s > \max s'$. Otherwise, by indiscernibility, every two members of K were equal.

Since the members of I are indiscernible, and by our assumption on K , we have that for every $\beta_0 < \beta_1 < \dots < \beta_{(m-k) \cdot n-1}$ in I if we let

$$b_i = f_\star(r^\sim \langle \beta_{(m-k)i}, \beta_{(m-k)i+1}, \dots, \beta_{(m-k)(i+1)-1} \rangle)$$

then $\psi(b_0, \dots, b_{n-1})$. This provides a set of cardinality κ in \mathcal{A} which is a ψ -cube. \square

3.3. $Q^{<\omega}$ reflection at successor cardinals. In this subsection we will discuss some cases in which the one can force the $Q^{<\omega}$ -reflection at successors to regular cardinals, starting from large cardinals at the level of huge cardinal.

We will start with the following technical definition:

Definition 18. Let \mathbb{P} be a forcing notion. We say that \mathbb{P} is κ - $Q^{<\omega}$ preserving if for every algebra of cardinality κ , \mathcal{A} , $Q^{<\omega}$ -formula φ , and $G \subseteq \mathbb{P}$ a generic filter

$$V \models \mathcal{A} \models \varphi \iff V[G] \models \mathcal{A} \models \varphi.$$

The class of κ - $Q^{<\omega}$ preserving forcings is closed under finite iterations. κ -closed forcing notion is κ - $Q^{<\omega}$ preserving. The same holds for κ -Knaster, well-met forcings. κ -c.c. forcing notions may not be κ - $Q^{<\omega}$ (e.g. a forcing that adds a branch to a κ -Suslin tree does not preserve the $Q^{<\omega}$ sentence "there is no set of cardinality κ of incompatible elements"). We remark that if there is a projection from \mathbb{P} onto \mathbb{Q} and \mathbb{P} is κ - $Q^{<\omega}$ preserving then \mathbb{Q} is also κ - $Q^{<\omega}$ preserving.

A κ - $Q^{<\omega}$ preserving forcing notion does not collapse κ . Otherwise, if $|\kappa| = \mu$ in the generic extension, for some $\mu < \kappa$, then the truth value of the $Q^{<\omega}$ formula $Q^1 x$, $x < \mu$ in the model $\langle \kappa, \leq \rangle$ was changed.

For the next theorem we would like to have a forcing notion that collapses cardinals below a Mahlo cardinal and behaves nicely under iterations and elementary embeddings. There are several such forcing notions in the literature (see [6], [4], [2], [11] and others).

For our results, in order to keep this paper as self-contained as possible, we will use a simple version of the forcing notion that was defined in [4].

Definition 19. Let $\mu < \kappa$ be regular cardinals. We denote by $\mathbb{EC}(\mu, < \kappa)$ the Easton support product $\prod_{\mu \leq \alpha < \kappa} \text{Col}(\alpha, < \kappa)$, where the product ranges over regular cardinals.

Let $\mathbb{P} = \mathbb{EC}(\mu, < \kappa)$. \mathbb{P} is the set of all partial functions such that:

- (1) $\text{dom}(f) \subseteq \{ \langle \alpha, \beta, \gamma \rangle \mid \mu \leq \alpha < \kappa, \beta \in [\alpha, \kappa) \text{ regular cardinals}, \gamma < \alpha \}$.
- (2) $f(\alpha, \beta, \gamma) \in \beta$.
- (3) $|\{ \alpha \mid \exists \langle \alpha, \beta, \gamma \rangle \in \text{dom}(f) \} \cap \rho| < \rho$ for all inaccessible ρ .
- (4) For all $\alpha < \kappa$, $|\{ \langle \beta, \gamma \rangle \mid \langle \alpha, \beta, \gamma \rangle \in \text{dom}(f) \}| < \alpha$.

Lemma 20. Let $\mu < \kappa$ be regular cardinals and assume that κ is Mahlo.

- (1) \mathbb{P} is κ -Knaster and μ -closed.
- (2) \mathbb{P} collapses every cardinal between μ and κ .

Proof. Let $\{p_i \mid i < \kappa\}$ be a sequence of conditions in \mathbb{P} . Let

$$g(i) = \sup\{\alpha, \beta, \gamma, p_i(\alpha, \beta, \gamma) \mid \langle \alpha, \beta, \gamma \rangle \in \text{dom}(p_i)\}.$$

For all $i < \kappa$, $g(i) < \kappa$ and therefore there is a club $C \subseteq \kappa$ of cardinals such that $\forall \rho \in C, \sup g''(\rho) \leq \rho$. Let $\{\rho_i \mid i < \kappa\}$ be an increasing enumeration of all the strongly inaccessible cardinals in C . Note that this is a stationary subset of κ .

Let $q_i = p_i \restriction [\rho_i, \kappa) \times [\rho_i, \kappa)$ namely the function p_i restricted to inputs of the form $\langle \alpha, \beta, \gamma \rangle$ where $\rho_i \leq \alpha, \beta$. By the definition of C , for every $i < j$, q_i and q_j are compatible, since their domain are disjoint - the domain of q_i is a subset of $\rho_{i+1} \times \rho_{i+1} \times \rho_{i+1}$ while the domain of q_i does not contain any triplet of the form $\langle \alpha, \beta, \gamma \rangle$ with $\alpha, \beta < \rho_j$.

Let us look at $r_i = p_i \upharpoonright \rho_i \times \rho_i$. Since ρ_i is strongly inaccessible, $r_i \in V_{\rho_i}$ (its domain is bounded below ρ_i). By fixing some enumeration of V_κ that maps elements of V_ρ to ordinals below ρ for every inaccessible ρ , the function $\rho_i \rightarrow r_i$ is equivalent to a regressive function of a stationary set. Therefore, by Fodor's lemma, there is a stationary subset S , such that for all $\rho_i, \rho_j \in S$, $r_i = r_j$. We conclude that for every $\rho_i, \rho_j \in S$, p_i is compatible with p_j . \square

Theorem 21. *Let $\mu < \kappa \leq \lambda < \delta$ be regular cardinals and assume that there is an elementary embedding $j: V \rightarrow M$ with $j(\kappa) = \delta$, $M^{j(\lambda)} \subseteq M$.*

Let $\mathbb{P} = \text{EC}(\mu, < \kappa)$ with Easton support and let $\mathbb{Q} = \text{EC}(\lambda, < \delta)^{V^\mathbb{P}}$.

*Then $V^{\mathbb{P} * \mathbb{Q}} \models j(\lambda) \xrightarrow[Q^{<\omega}]{} \lambda$.*

Proof. By Lemma 20, \mathbb{P} is κ -Knaster and \mathbb{Q} is forced to be δ -Knaster in $V^\mathbb{P}$.

Let \mathbb{N} be the termspace forcing for \mathbb{Q} and let \mathbb{R} be $\text{EC}(\lambda, < \delta)^V$.

Lemma 22. *There is a projection from \mathbb{R} onto \mathbb{N} .*

Proof. Using the fact that κ is huge cardinal and in particular δ -supercompact, for every regular cardinal, $\rho \geq \kappa$, $\rho^{<\kappa} = \rho$. In particular, we can construct a bijection between all the nice \mathbb{P} -names of ordinals below a regular ρ and ρ (using the fact that \mathbb{P} is κ -c.c.). Using this bijection we can identify a condition in \mathbb{R} as a partial function to names of ordinals in the appropriate domain and get a projection. \square

Let $j: V \rightarrow M$ be an elementary embedding, witnessing the hugeness of κ , $j(\kappa) = \delta$, $j''\lambda \in M$. $j(\mathbb{P} * \mathbb{Q}) = j(\mathbb{P}) * j(\mathbb{Q})$. We want to show that one can find a weak master condition for this forcing.

$j(\mathbb{P}) = \prod_{\mu \leq \alpha < \delta} \text{Col}(\alpha, < \delta)$. Clearly, \mathbb{P} embeds into $j(\mathbb{P})$. Moreover, \mathbb{R} also embeds into $j(\mathbb{P})$ (and into disjoint set of coordinates) and thus after forcing with $j(\mathbb{P})$ we have a generic filter $G \subset \mathbb{P}$. Using Silver criteria for extending elementary embeddings to generic extension, one can extend j by forcing with $j(\mathbb{P}) * j(\mathbb{Q})/\mathbb{P}$. $j(\mathbb{Q})$ is highly directed closed (at least δ -closed). In fact, if $\lambda > \kappa$, one can find in V a generic for this part. Thus, $j(\mathbb{P}) * j(\mathbb{Q})/\mathbb{P}$ is δ - $Q^{<\omega}$ preserving (since $j(\mathbb{P})$ and $j(\mathbb{Q})$ are).

Let \mathcal{A} be an algebra on $j(\lambda)$. Let us extend j to an embedding \tilde{j} by forcing with $j(\mathbb{P} * \mathbb{Q})/(\mathbb{P} * \mathbb{Q})$. In $M[j(G * H)]$, $\tilde{j}''\mathcal{A}$ is an elementary substructure of $j(\mathcal{A})$ of cardinality $j(\lambda)$. We want to show that it is $Q^{<\omega}$ -elementary. Since the forcing $j(\mathbb{P} * \mathbb{Q})/(\mathbb{P} * \mathbb{Q})$ is $j(\lambda)$ - $Q^{<\omega}$ preserving, every $Q^{<\omega}$ formula that holds in $\tilde{j}''\mathcal{A}$ (and hence in \mathcal{A}) in $M[j(G)][j(H)]$, holds in \mathcal{A} in $V[G][H]$ as well. Therefore, it holds in $j(\mathcal{A})$ - as wanted. \square

Taking $\lambda = \kappa$ and μ regular we obtain $\mu^{++} \xrightarrow[Q^{<\omega}]{} \mu^+$. The proof shows that the result holds also for languages of cardinality $< \kappa$, so we can write $\mu^{++} \xrightarrow[Q^{<\omega}]{} \mu^+$. Assuming that GCH holds in the ground model, it also holds in the generic extension.

Corollary 23. *It is consistent, relative to a huge cardinal, that $\aleph_3 \xrightarrow[Q^{<\omega}]{} \aleph_2$ and GCH holds.*

Using $\lambda = \kappa^{+\omega+1}$ in order to obtain a gap we can get:

Corollary 24. *It is consistent, relative to 2-huge cardinal, that $\aleph_{\omega \cdot 2+1} \xrightarrow[Q^{<\omega}]{} \aleph_{\omega+1}$.*

3.4. MM reflection to \aleph_1 . In the subsection we will show how to derive instances of $Q^{<\omega}$ -reflection from some cardinal to \aleph_1 using a sufficiently good ideal. For a survey about ideals and their connection to large cardinals, see [3]. In particular we will obtain the consistency of $\aleph_2 \xrightarrow[Q^{<\omega}]{} \aleph_1$ from a measurable cardinal.

Let \mathcal{I} be an ideal on κ such that:

- (1) $\{\alpha\} \in \mathcal{I}$ for all $\alpha < \kappa$.
- (2) \mathcal{I} is κ -complete.
- (3) The forcing that adds a generic ultrafilter to \mathcal{P}/\mathcal{I} is $\omega + 1$ -strategically closed.

Theorem 25. *Assume that there is ideal \mathcal{I} as above and $\diamond(\omega_1)$. Then $\kappa \xrightarrow[Q^{<\omega}]{} \omega_1$.*

Proof. The proof follows closely the proof of the completeness theorem for the logic $\mathcal{L}(Q^{<\omega})$ - the first order logic extended by the Magidor-Malitz quantifier. This result requires $\diamond(\omega_1)$ as well. See [5], section 7.3.

We start with a countable model M_0 , and repeatedly add new elements to it. At each step we essentially enlarge M_α by adding some ordinal $\zeta < \kappa$ which is generic over M_α for the forcing $\mathcal{P}(\kappa)/\mathcal{I}$ (i.e. $\{A \in M \mid \zeta \in A\}$ is a M_α -generic ultrafilter). Eventually, we will show that we can arrange the limit model M_{ω_1} to be a $Q^{<\omega}$ elementary submodel of some elementary substructure of $H(\chi)$ (χ large enough) of cardinality κ that contains all ordinals below κ . Note that this is the general case, as for any algebra \mathcal{A} on κ , we may assume that $\mathcal{A} \in M_0$.

Throughout the rest of the proof, χ is a regular cardinal above 2^{2^κ} .

Lemma 26. *Let M be a countable elementary substructure of $H(\chi)$. There is $\zeta < \kappa$ such that $\{A \in M \mid \zeta \in A\}$ is M -generic for the forcing $(\mathcal{P}(\kappa)/\mathcal{I})^M$. Moreover, $M^* := \{f(\zeta) \mid f: \kappa \rightarrow V, f \in M\}$ is a proper extension of M and $M^* \prec H(\chi)$.*

Proof. Let $\{I_n \mid n < \omega\}$ list all the maximal antichains of the forcing $(\mathcal{P}(\kappa)/\mathcal{I})^M$ in M . Since the positive sets forcing of \mathcal{I} is σ -strategically closed, there is a sequence of \mathcal{I} -positive sets $\langle A_n \mid n < \omega \rangle$ such that $A_{n+1} \subseteq A_n$, $A_n \in I_n$ and $\bigcap_{n < \omega} A_n \notin \mathcal{I}$. Any $\zeta \in \bigcap A_n$ will generate a M -generic filter.

Since for every $x \in M$ the constant function $c_x(\alpha) = x$ is in M , $M \subseteq M^*$. Since the identity function $id(\alpha) = \alpha$ is in M , $\zeta \in M^*$ so M^* is strictly larger than M .

In order to show that $M^* \prec H(\chi)$ we use Tarski-Vaught criterion. Let $\varphi(x, b)$ be a formula with $b \in M^*$, and assume $H(\chi) \models \exists x \varphi(x, b)$. We need to show that $M^* \models \exists x \varphi(x, b)$. $b = g(\zeta)$ for some $g \in M$. Let

$$B = \{\alpha < \kappa \mid H(\chi) \models \exists x \varphi(x, g(\alpha))\} \notin \mathcal{I}.$$

This set is definable and therefore it is a member of M . Thus, applying the axiom of choice inside of M there is a function $f \in M$ that assign to every element $\alpha \in B$ a witness x such that $\varphi(x, g(\alpha))$. In M ,

$$B = \{\alpha < \kappa \mid \varphi(f(\alpha), g(\alpha))\}$$

and by elementarity, the same holds in V . Since $\zeta \in B$, $\varphi(f(\zeta), g(\zeta))$, so $f(\zeta)$ witnesses $M^* \models \exists x \varphi(x, b)$. \square

Let us define a sequence of models. $M_0 = M$, $M_{\alpha+1} = M_\alpha^*$ (we will define the M_α -generic filters more explicitly in the course of the proof). For limit ordinal $\beta \leq \omega_1$, $M_\beta = \bigcup_{\alpha < \beta} M_\alpha$.

We would like to get that M_{ω_1} witnesses an instance of $\kappa \xrightarrow[Q^{<\omega}]{} \omega_1$. In order to achieve this, during the iteration we will pick the generic elements in a way that will handle any potential counterexample for the reflection $M_{\omega_1} \cap \kappa \prec_{Q^{<\omega}} H(\chi) \cap \kappa$.

Let $\phi(x)$ be a formula (with parameters from M) and let A be an \mathcal{I} -positive set. we define:

$$\partial_A \phi(w) := \{ \alpha \in A \mid \neg \phi(w(\alpha)) \} \in \mathcal{I}.$$

For a type $\Phi(x)$ with parameters in M , (not necessary in M), we define

$$\partial_A \Phi(w) = \{ \partial_A \phi \mid \phi(x) \in \Phi(x) \}.$$

These types control which types will be omitted in the next step of the construction. If w realized $\partial_A \Phi$ in M , then for every choice of $\zeta \in A$, except an \mathcal{I} -null set, Φ is realized in M^* . On the other hand, if for every $\zeta \in A$, Φ is realized in M^* (where it is defined using ζ) then there is some positive set $B \subseteq A$ such that $\partial_B \Phi$ is realized. Otherwise, we could remove, outside of M , an \mathcal{I} -null set and verify that this is not the case. This process cannot be done inside of M , since in general $\Phi \notin M$.

One can repeat this process countably many times (using the strategically closure of the forcing) and verify that for a countable set of types $\{\Phi_n(x) \mid n < \omega\}$ if $\partial_A \Phi_n$ is omitted in M for all $n < \omega$ and $A \in M$ then $\Phi_n(x)$ is omitted in M^* .

In M , there are names for a positive sets in M^* . Those are essentially the functions $f: \kappa \rightarrow \mathcal{I}^+$ that appear in M . One can define, for a given formula φ , a positive set A and a name of a positive set \dot{B} the formula $\partial_A \partial_{\dot{B}} \varphi$, in the natural way. We define $\partial_A \partial_{\dot{B}} \Phi$ for a type Φ accordingly. We can continue this way and define the derivative of a type relative to any finite sequence of names of positive sets in the iterated forcing (in the narrow sense: the m -th set \dot{B} is a function from κ^m to the positive sets).

Let us enrich the language of set theory by all the members of M (as constants).

Lemma 27. *Let φ be a formula with k free variables. Let $Z \subseteq M$ be a maximal φ -cube. Let Φ be the type*

$$\{ \psi(x) \mid \forall a \in Z, \psi(a) \} \cup \{ x \neq a \mid a \in Z \}.$$

If $V \models \neg Q^k \varphi$ then M omits all the derivatives of Φ .

Proof. Φ contains the formulas $\varphi(a_0, \dots, a_{k-2}, x)$ for all $a_i \in Z$ and therefore M does not realize Φ , by the maximality of Z .

Assume that M realizes $\partial_{A_0} \partial_{\dot{A}_1} \dots \partial_{\dot{A}_{m-1}} \Phi$ for some $A_0, \dots, A_{m-1} \in M$. So there is some $b \in M$ such that:

$$\forall^* \alpha_0 \in A_0 \forall^* \alpha_1 \in A_1 (\alpha_0) \dots \forall^* \alpha_{m-1} \in A_{m-1} (\alpha_0, \dots, \alpha_{m-2}) \psi(b(\alpha_{m-1}))$$

for every $\psi \in \Phi$.

We may assume that for all $x \in M$, $b^{-1}(\{x\}) \in \mathcal{I}$. For all relevant ordinal (one which escape all the \mathcal{I} -null set in the quantifiers), this is true by the maximality of Z and the fact that b is "forced" to be different than all members of Z in M . We can complete the rest of the values (which are essentially elements outside the sets in range A_i , $i < m$ and A_0) with dummy values.

Let us denote by $\forall^* \alpha \varphi(\alpha)$ the assertion that $\{ \alpha \mid \neg \varphi(\alpha) \} \in \mathcal{I}$.

Taking $\psi(x)$ to be $\varphi(a_0, \dots, a_{k-2}, x)$ (and omitting the evaluations in the \dot{A}_i) we get:

$$\forall^* \alpha_0 \in A_0 \forall^* \alpha_1 \in \dot{A}_1 \dots \forall^* \alpha_{m-1} \in \dot{A}_{m-1} \varphi(a_0, \dots, a_{k-2}, b(\alpha_{m-1}))$$

By the definition of Ψ , replacing a_{k-2} by the variable x , we obtain a formula in Ψ . So we conclude that:

$$\begin{aligned} \forall^* \alpha_0 \in A_0 \forall^* \alpha_1 \in \dot{A}_1 \dots \forall^* \alpha_{m-1} \in \dot{A}_{m-1} \\ \forall^* \beta_0 \in A_0 \forall^* \beta_1 \in \dot{A}_1 \dots \forall^* \beta_{m-1} \in \dot{A}_{m-1} \\ \varphi(a_0, \dots, a_{k-3}, b(\beta_{m-1}), b(\alpha_{m-1})) \end{aligned}$$

Repeating this process and relabelling:

$$\begin{aligned} \forall^* \alpha_0^0 \in A_0 \forall^* \alpha_1^0 \in \dot{A}_1 \cdots \forall^* \alpha_{m-1}^0 \in \dot{A}_{m-1} \\ \forall^* \alpha_0^1 \in A_0 \forall^* \alpha_1^1 \in \dot{A}_1 \cdots \forall^* \alpha_{m-1}^1 \in \dot{A}_{m-1} \\ \vdots \\ \forall^* \alpha_0^{k-1} \in A_0 \forall^* \alpha_1^{k-1} \in \dot{A}_1 \cdots \forall^* \alpha_{m-1}^{k-1} \in \dot{A}_{m-1} \\ \varphi(b(\alpha_{m-1}^0), b(\alpha_{m-1}^1), \dots, b(\alpha_{m-1}^{k-1})) \end{aligned}$$

Let us look on this last formula (which is true in M) and let us say that a set D is *solid* iff for all $a_0, \dots, a_r \in D$ ($0 \leq r \leq k$),

$$\begin{aligned} \forall^* \alpha_0^0 \in A_0 \forall^* \alpha_1^0 \in \dot{A}_1 \cdots \forall^* \alpha_{m-1}^0 \in \dot{A}_{m-1} \\ \forall^* \alpha_0^1 \in A_0 \forall^* \alpha_1^1 \in \dot{A}_1 \cdots \forall^* \alpha_{m-1}^1 \in \dot{A}_{m-1} \\ \vdots \\ \forall^* \alpha_0^{r-k-1} \in A_0 \forall^* \alpha_1^{r-k-1} \in \dot{A}_1 \cdots \forall^* \alpha_{m-1}^{r-k-1} \in \dot{A}_{m-1} \\ \varphi(a_0, \dots, a_{r-1}, b(\alpha_{m-1}^0), b(\alpha_{m-1}^1), \dots, b(\alpha_{m-1}^{k-r-1})) \end{aligned}$$

The empty set is solid, so by using Zorn's lemma in M , we can find a maximal solid set, $D \in M$.

Lemma 28. $M \models |D| = \kappa$.

Proof. Assume otherwise. We will find $c \in M$ and outside D such that $\{c\} \cup D$ is solid. If $b^{-1}(D)$ was \mathcal{I} -positive then there must have been some $d \in D$ such that $b^{-1}(\{d\})$ is \mathcal{I} -positive, and we assumed that this is not the case.

Let us iteratively narrow down, in V , the positive sets A_0, \dot{A}_1, \dots and replace the quantifier \forall^* by \forall . We would still remain with a positive sets. Moreover, we may assume that all of them are disjoint from $b^{-1}(D)$. Pick any $\alpha_0 \in A_0$, $\alpha_1 \in \dot{A}_1(\alpha_0), \dots, \alpha_{m-1} \in \dot{A}_{m-1}(\alpha_0, \dots, \alpha_{m-2})$. Let $c = b(\alpha_n)$. $c \notin D$ and for every $a_i \in D$, $\varphi(a_0, \dots, a_{k-2}, c)$, as wanted. \square

We conclude that D is a φ -cube of cardinality κ . But by elementarity, D is a φ -cube in V as well. \square

In order to finish the proof we need to explain how to choose the sets Z . Here the diamond comes into the picture. Let $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ a $\diamond(\omega_1)$ sequence. For convenience, we will assume that S_α is a pair (A_α, ϕ_α) where $A_\alpha \subseteq \alpha$ and ϕ_α is a $Q^{<\omega}$ formula $Q^n \varphi$ with parameters in α .

For every $i < \omega_1$, let us choose a bijection between $M_{i+1} \setminus M_i$ and $\omega \cdot (i+1) \setminus \omega \cdot i$. Connecting those bijections we obtain a continuous bijection between M_{ω_1} and ω_1 .

For every α , if A_α is a maximal φ_α -cube in M_α , we define Φ_i to be the type which was defined in lemma 27. Otherwise, we do nothing.

When enlarging M_i to be M_{i+1} we omit the types $\{\Phi_j \mid j \leq i\}$. Let $\psi = Q^n \varphi$ be a $Q^{<\omega}$ formula. Assume that M_{ω_1} satisfies ψ and that Z is a maximal φ -cube. Then on club many points, $Z \cap \alpha$ is a maximal φ -cube. Therefore, there is a point $\alpha < \omega_1$ such that $Z \cap \alpha = A_\alpha$, $\varphi_\alpha = \varphi$. But the corresponding type was not omitted, since it was enlarged, so $V \models \psi$, as needed. \square

We remark that for successor of regular κ , the existence of such an ideal \mathcal{I} is equiconsistent with a measurable cardinal. Unfortunately, for successor of singular cardinal, such ideal cannot exist.

Question. Is $\aleph_{\omega+1} \xrightarrow{Q^{<\omega}} \aleph_1$ consistent?

REFERENCES

1. C. C. Chang and H. J. Keisler, *Model theory*, third ed., Studies in Logic and the Foundations of Mathematics, vol. 73, North-Holland Publishing Co., Amsterdam, 1990. MR 1059055 (91c:03026)
2. Monroe Blake Eskew, *Measurability properties on small cardinals dissertation*, Ph.D. thesis, UNIVERSITY OF CALIFORNIA, IRVINE, 2014.
3. Matthew Foreman, *Ideals and generic elementary embeddings*, Handbook of set theory, Springer, 2010, pp. 885–1147.
4. Matthew Foreman and Richard Laver, *Some downwards transfer properties for \aleph_2* , Adv. in Math. **67** (1988), no. 2, 230–238. MR 925267 (89h:03090)
5. Wilfrid Hodges, *Building models by games*, London Mathematical Society Student Texts, vol. 2, Cambridge University Press, Cambridge, 1985. MR 812274 (87h:03045)
6. Kenneth Kunen, *Saturated ideals*, J. Symbolic Logic **43** (1978), no. 1, 65–76. MR 495118 (80a:03068)
7. Menachem Magidor, *On the role of supercompact and extendible cardinals in logic*, Israel J. Math. **10** (1971), 147–157. MR 0295904 (45 #4966)
8. Menachem Magidor and Jerome Malitz, *Compact extensions of $L(Q)$. Ia*, Ann. Math. Logic **11** (1977), no. 2, 217–261. MR 0453484 (56 #11746)
9. Itay Neeman and John Steel, *Equiconsistencies at subcompact cardinals*, submitted.
10. Assaf Rinot, *Chain conditions of products, and weakly compact cardinals*, Bull. Symb. Log. **20** (2014), no. 3, 293–314. MR 3271280
11. Masahiro Shioya, *The Easton collapse and a saturated filter*, (2011).